Small clones and the projection property

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Abstract

In 1986, the second author classified the minimal clones on a finite universe into five types. We extend this classification to infinite universes and to multiclones. We show that every non-trivial clone contains a "small" clone of one of the five types. From it we deduce, in part, an earlier result, namely that if $\mathcal{C}$ is a clone on a universe A with at least two elements, that contains all constant operations, then all binary idempotent operations are projections and some m-ary idempotent operation is not a projection some $m \geq 3$ if and only if there is a Boolean group G on A for which $\mathcal{C}$ is the set of all operations $f(x_1, \ldots, x_n)$ of the form $a + \sum_{i \in I} x_i$ for $a \in A$ and $I \subseteq \{1, \ldots, n\}$.

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1 Definitions, notations, results

Let A be a fixed universe of cardinality at least 2. Denote by $\mathfrak{P}(A)$ the family of subsets of A. Denote by $\mathbb{N}$ the set of non-negative integers and by $\mathbb{N}^*$ the set $\mathbb{N} \setminus \{0\}$. For $n \in \mathbb{N}^*$ a map $f : A^n \to \mathfrak{P}(A)$ is an *n-ary multioperation* on A. Call f an *hyperoperation* if the empty set is not in imf, the image of f. Special binary hyperoperations, called *hypergroups*, where introduced in 1934 [5] and there is a sizeable literature on them, see [1], [2]. If $imf \subseteq \{\{b\} : b \in A\} \cup \{\emptyset\}$, the map f is called a *partial operation* on A. If we identify each singleton value $\{b\}$ of f with the element b and declare that f is not defined at all $a \in A^n$ with $f(a) = \emptyset$ then, clearly, f becomes a partial operation in the usual sense. If f is such that $imf \subseteq \{\{b\} : b \in A\}$ the same identification of all singleton values yields an operation on A in the usual sense. Denote by $\mathcal{M}^{(n)}$, $\mathcal{H}^{(n)}$, $\mathcal{P}^{(n)}$ and $\mathcal{O}^{(n)}$ the set of n-ary multioperations, hyperoperations, partial operations and operations on A. Set $\mathcal{M} := \bigcup\{\mathcal{M}^{(n)} : n \in \mathbb{N}^*\}$ and define similarly $\mathcal{H}, \mathcal{P}$ and $\mathcal{O}$. For a subset $\mathcal{Z}$ of $\mathcal{M}, \mathcal{H}, \mathcal{P}$ and $\mathcal{O}$, call the pair $< A; \mathcal{Z} >$ a (nonindexed) *multialgebra, hyperalgebra, partial algebra* and *algebra* on A.

For $\mathcal{C} \subseteq \mathcal{M}$ and $n \in \mathbb{N}^*$ set $\mathcal{C}^{(n)} := \mathcal{C} \cap \mathcal{M}^{(n)}$. In the sequel $\approx$ denotes an identity on A (i.e. the two sides are equal for all values of the variables in A) and we write $:\approx$ for a defining identity. For $f \in \mathcal{M}^{(n)}$ and π a permutation of the set $\{1, \ldots, n\}$, the operation f_π defined by $f_\pi(x_1, \ldots, x_n) :\approx f(\pi(x_1), \ldots, \pi(x_n))$ is an *isomer* of f. A multioperation $f \in \mathcal{M}$ is *idempotent* provided $f(x, \ldots, x) \approx \{x\}$. We denote by $\mathcal{I}$ the set of all idempotent multioperations on A. For $n = 1, 2, 3$ the multioperations from $\mathcal{M}^{(n)}$ are called, respectively, *unary*, *binary* and *ternary*. A ternary multioperation f is a *majority* multioperation if $f(x, x, y) \approx f(x, y, x) \approx f(y, x, x) \approx \{x\}$; if, on the opposite, $f(x, x, y) \approx f(x, y, x) \approx f(y, x, x) \approx \{y\}$ the multioperation f is a *minority* multioperation. If $f(x, x, y) \approx y \approx f(y, x, x)$ the multioperation f is a *Mal'tsev multioperation* and, if in addition $f(x, y, x) \approx \{x\}$, this is a *Pixley multioperation.* For $i, n \in \mathbb{N}^*$ with $i \leq n$, define the *i-th n-ary projection* e^n_i by setting $e^n_i(x_1, \ldots, x_n) :\approx x_i$. Set $\mathcal{Q} := \{e^n_i : i, n \in \mathbb{N}^*\}$. For $n \geq 3$ and $1 \leq i \leq n$, call $f \in \mathcal{M}^{(n)}$ a *semiprojection on its i-th coordinate* if $f(a_1, \ldots, a_n) = \{a_i\}$ whenever $a_1, \ldots, a_n \in A$ are not pairwise distinct.

On the set $\mathcal{M}$ we define a countable set $\{\pi_{ij} : i, j \in \mathbb{N}^*\}$ of partial operations. For $i, j \in \mathbb{N}^*$, the map $\pi_{ij} : \mathcal{M}^{(i)} \times (\mathcal{M}^{(j)})^i \to \mathcal{M}^{(i)}$ is defined as follows. Let $f \in \mathcal{M}^{(i)}$ and $(g_1, \ldots, g_i) \in (\mathcal{M}(j))^i$. For every $a := (a_1, \ldots, a_j) \in A^j$ set:

$$\pi_{ij}(f, g_1, \ldots, g_i)(a) := \bigcup\{f(u_1, \ldots, u_i) : u_1 \in g_1(a), \ldots, u_i \in g_i(a)\} \quad (1)$$

Notice that (1) makes sense since $g_1, \ldots, g_i)(a)$ and $f(u_1, \ldots, u_i)$ are subsets of A. Also, if $f, g_1, \ldots, g_i$ are operations, the right-hand side of (1) is the standard $f(g_1(a), \ldots, g_i(a))$.

For $\mathcal{Z} \subseteq \mathcal{M}$ denote by $[\mathcal{Z}]$ the least member of $\mathcal{M}$ containing $\mathcal{Z} \cup \mathcal{Q}$ and closed under all π_{ij}, with $i, j \in \mathbb{N}^*$. The subsets of $\mathcal{M}$ of the form $[\mathcal{Z}]$ for some $\mathcal{Z} \subseteq \mathcal{M}$ are called *multiclones*. If $\mathcal{Z} \subseteq \mathcal{X}$ for $\mathcal{X} \in \{\mathcal{H}, \mathcal{P}, \mathcal{O}\}$, then $[\mathcal{Z}]$ is said to be, respectively, a *hyperclone*, *partial clone* and *clone*. For $\mathcal{Z} \subseteq \mathcal{O}$ the clone $[\mathcal{Z}]$ is the set of term operations of the algebra $< A, \mathcal{Z} >$; this fact can be extended to $\mathcal{Z} \subseteq \mathcal{X} \in \{\mathcal{M}, \mathcal{P}, \mathcal{H}\}$.

A multiclone is said to be *minimal* if $\mathcal{Q}$ is its only proper submulticlone. An operation f is said to be *minimal* if the multiclone $[f]$ generated by f is minimal and f is of minimum arity among the multioperations in $[f] \backslash \mathcal{Q}$. For an example, let $f_n \in \mathcal{M}^{(n)}$ with $f_n(a) := \emptyset$ for all $a \in A^n$. Then $\mathcal{Q} \cup \{f_n : n \in \mathbb{N}^*\}$ is a minimal multiclone and f_1 is minimal. It is known that on a finite universe, every clone distinct from $\mathcal{Q}$ contains a minimal clone and, as shown by the second author [9], the minimal operations fall into five distinct types. Despite the fact that on an infinite universe a clone distinct from $\mathcal{Q}$ does not necessarily contain a minimal clone, it turns out that the main feature of the second author's result is preserved: clones, and particularly minimal ones, can be classified by means of the five types of operations, Á. Szendrei [11]. It was mentioned by J. Pantović and G.Vojvodic [6] that on a finite universe, every hyperclone distinct from $\mathcal{Q}$ contains a minimal hyperclone of one of the five types. Here we show first that the classification into five types extends to multiclones on an arbitrary universe.

In order to present our result, we recall that a *Boolean group* is a 2-elementary group $< A; +, 0 >$, that is a group with neutral element 0 satisfying $a + a = 0$ for all $a \in A$. It is well known that a Boolean group is necessarily abelian; in fact such a group on A finite exists exactly if $|A|$ is a power of 2 and in that case the group is isomorphic to a power of $\mathbb{Z}/2\mathbb{Z}$.

Theorem 1.1 *Let $\mathcal{C}$ be a multiclone on A. If $\mathcal{C} \neq \mathcal{Q}$, then $\mathcal{C} \setminus \mathcal{Q}$ contains either:*
1) *a unary multioperation;*
2) *a binary idempotent multioperation;*
3) *a majority multioperation;*
4) *a semiprojection;*
5) *the term operation $x + y + z$ of a Boolean group $< A; +, 0 >$.*

Next, we apply our result to the projection property, a property we introduced in [7] for structures of various sorts, like posets, graphs, metric spaces.

A structure R is *n-projective* if the only idempotent morphisms from its n-power R^n into R are the projections. We looked at the relationship between these properties for various values of n. One of our results (Theorem 1.1 of [7], see also [8]) can be deduced, in part, from Theorem 1.1

Theorem 1.2 *The following are equivalent for a clone $\mathcal{C}$ on a universe A with at least two elements.*
(i) *$\mathcal{C}$ contains all constant operations, all its binary idempotent operations are projections while some n-ary idempotent operation is not a projection;*
(ii) *there is a Boolean group G on A for which $\mathcal{C}$ is the set F_G of all operations of the form $f(x_1, \ldots, x_n) \approx a + \sum_{i \in I} x_i$ for $a \in A$, $I \subseteq \{1, \ldots, n\}$ and $n \in \mathbb{N}^*$.*

A result similar to Theorem 1.2 was obtained independently in [3] (Lemma 2.6).

2 Proof of Theorem 1.1

Our proof is an adaptation of the proof from [9] (see also [7]).

Let n be the least positive integer such that $\mathcal{C}^{(n)} \neq \mathcal{Q}^{(n)}$. Notice that for $n > 1$, $f \in \mathcal{C}^{(n)}$ and $1 \leq i < j \leq n$, the multioperation $g(x_1, \ldots, x_{n-1}) :\approx f(x_1, \ldots, x_i, \ldots, x_{j-1}, x_i, x_j, \ldots, x_{n-1})$ belongs to $\mathcal{C}^{(n-1)}$ and hence is a projection. If $n = 1, 2$ then any $f \in \mathcal{C}^{(n)} \setminus \mathcal{Q}^{(n)}$ satisfies 1) or 2) of Theorem 1.1. If $n \geq 4$ then, according to the following lemma, any $f \in \mathcal{C}^{(n)} \setminus \mathcal{Q}^{(n)}$ satisfies 4).

Lemma 2.1 *Let $n, n \geq 4$, and $f \in \mathcal{C}^{(n)}$; suppose that every multioperation obtained from f by identifying two variables is a projection, then f is a semiprojection.*

Proof. We assume $|f(a)| = 1$ for all $a := (a_1, \ldots, a_n) \in A^n$ such that $a_1, \ldots, a_n$ are not pairwise distinct. We can then apply the well-known Swierczkowski Lemma[10]. Indeed, as it turns out, the fact that we may have $|f(a)| \neq 1$, for some $a := (a_1, \ldots, a_n) \in A^n$ with $a_1, \ldots a_n$ pairwise distinct, is irrelevant. □

Thus, we may suppose that $n = 3$. In this case, the following lemma asserts that we may find f in case 3, 4 or 5, which proves Theorem 1.1.

Lemma 2.2 *Suppose $\mathcal{C}^{(2)} = \mathcal{Q}^{(2)}$ and $\mathcal{C}^{(3)} \setminus \mathcal{Q}^{(3)} \neq \emptyset$. If $\mathcal{C}^{(3)} \setminus \mathcal{Q}^{(3)}$ contains no semiprojection and no majority multioperation then $\mathcal{C}^{(3)} \setminus \mathcal{Q}^{(3)} = \{m\}$,*

where m is a totally symmetric minority multioperation. If, moreover, $\mathcal{C}^{(4)} \setminus \mathcal{Q}^{(4)}$ contains no semiprojection then m is the term operation $x+y+z$ of a Boolean group $< A; +, 0 >$.

Proof.

Let $f \in \mathcal{C}^{(3)} \setminus \mathcal{Q}^{(3)}$. As every multioperation obtained from f by identifying two variables is a projection, there are $a, b, c \in \{1, 2\}$ such that

$$f(x_1, x_1, x_2) \approx x_a, \quad f(x_1, x_2, x_1) \approx x_b, \quad f(x_1, x_2, x_2) \approx x_c \tag{2}$$

We denote by χ_f the ordered triple abc and abreviate (1) by:

$$\overline{f}(112) = a, \quad \overline{f}(121) = b, \quad \overline{f}(122) = c. \tag{3}$$

Thus, if $\chi_f \in \{111, 122, 212\}$ then f is a semiprojection; if $\chi_f = 112$ then f is a majority multioperation; if $\chi_f = 221$ then f is a minority multioperation, and if $\chi_f = 211$ then f is a Pixley multioperation.

Claim 2.3 *If $\chi_f \in \{121, 222\}$ then $\chi_h = 211$ for some $h \in \mathcal{C}^3$.*

Proof of Claim 2.3. If $\chi_f = 121$, set $h(x_1, x_2, x_3) :\approx f(x_1, x_3, x_2)$. Then

$$\overline{h}(112) = \overline{f}(121) = 2, \;\; \overline{h}(121) = \overline{f}(112) = 1, \;\; \overline{h}(122) = \overline{f}(122) = 1$$

proving $\chi_h = 211$. If $\chi_f = 222$, set $h(x_1, x_2, x_3) :\approx f(x_2, x_1, x_3)$. Then $\overline{h}(112) = \overline{h}(211) = \overline{h}(212) = 2$ and so $\overline{h}(112) = 2$, $\overline{h}(121) = 1$ and $\overline{h}(121) = 1$, proving $\chi_h = 211$ and the claim. □

As it is well known (see e.g. Theorem 9.3.2 p.201 [4]) a clone $\mathcal{C}$ contains a Pixley operation if and only if it contains a majority and a Mal'tsev operation; in fact, as shown in the proof of Lemma 2.7 [9] p. 413, this amounts to the fact that $\mathcal{C}$ contains a majority and a minority operation. This extends to multioperations. For reader's convenience, we reprove what we need.

Claim 2.4 *If $\chi_h = 211$ then $\mathcal{C}$ contains a majority multioperation.*

Proof of Claim 2.4. Set $m(x_1, x_2, x_3) :\approx h(x_1, h(x_1, x_2, x_3), x_3)$. In view of $\chi_h = 211$, we get:

$$\overline{m}(112) = \overline{h}(1\overline{h}(112)2) = \overline{h}(112) = 1,$$

$$\overline{m}(121) = \overline{h}(1\overline{h}(121)1) = \overline{h}(111) = 1,$$

$$\overline{m}(122) = \overline{h}(1\overline{h}(122)2) = \overline{h}(112) = 1\overline{h}(111) = 1.$$

proving $\chi_g = 221$ and the claim. $\square$

Supposing that $\mathcal{C}^{(2)} = \mathcal{Q}^{(2)}$ and that $\mathcal{C}^{(3)} \setminus \mathcal{Q}^{(3)}$ is non-empty and contains no semiprojection and no majority multioperation, it follows from Claim 2.3 and Claim 2.4 that $\mathcal{C}^{(3)} \setminus \mathcal{Q}^{(3)}$ contains only minority multioperations.

Claim 2.5 *Let $f_1, f_2 \in \mathcal{C}^{(3)}$ be minority multioperations. Then*

$$f_1(f_2(x_1, x_2, x_3), x_2, x_3) \approx x_1 \tag{4}$$

$$f_1(x_1, f_2(x_1, x_2, x_3), x_3) \approx x_2 \tag{5}$$

$$f_1(x_1, x_2, f_2(x_1, x_2, x_3)) \approx x_3 \tag{6}$$

Proof of Claim 2.5. Denote by $s(x_1, x_2, x_3)$ the left-hand side of (4). Then we have successively:

$$\overline{s}(112) = \overline{f}_1(\overline{f}_2(112)12) = \overline{f}_1(212) = 1,$$

$$\overline{s}(121) = \overline{f}_1(\overline{f}_2(121)21) = \overline{f}_1(221) = 1,$$

$$\overline{s}(122) = \overline{f}_1(\overline{f}_2(122)22) = \overline{f}_1(122) = 1.$$

Hence, s is a semiprojection. Since $s \in \mathcal{C}^{(3)}$ and $\mathcal{C}^{(3)} \setminus \mathcal{Q}^{(3)}$ contains no semiprojection, s is a projection, that is $s = e_1^3$ proving (4). Identities (5) and (6) follow from (4) applied to the isomers $f_1(x_2, x_1, x_3)$ and $f_1(x_2, x_3, x_1)$ of f. This proves the claim. $\square$

Claim 2.6 *Let $f \in \mathcal{C}$ be a minority multioperation and $a, b, c \in A$. Then $f(a, b, c) = \{a\}$ if and only if $b = c$.*

Proof of Claim 2.6. If $b = c$ then, since f is a minority, $f(a, b, c) = f(a, b, b) = \{a\}$. Conversely, suppose $f(a, b, c) = \{a\}$. From (5) of Claim 2.5 applied to $f_1 = f_2 := f$, we have $f(x_1, f(x_1, x_2, x_3), x_3) \approx x_2$, in particular $f(a, f(a, b, c), c) = \{b\}$. With $f(a, b, c) = \{a\}$, this gives $f(a, a, c) = \{b\}$. Since f is a minority, we get $b = c$, as claimed. $\square$

Claim 2.7 *$\mathcal{C}$ contains only one minority multioperation.*

Proof of Claim 2.7. Let $f_1, f_2 \in \mathcal{C}$ be minority multioperations. Set

$$h(x_1, x_2, x_3) :\approx f_1(x_1, f_1(x_1, x_2, x_3), f_2(x_1, x_2, x_3)).$$

Proceeding as in the above proof of (4) we obtain that h is a semiprojection on the first coordinate, hence the first projection, that is

$$f_1(x_1, f_1(x_1, x_2, x_3), f_2(x_1, x_2, x_3)) \approx x_1.$$

According to Claim 2.6

$$f_1(x_1, x_2, x_3) \approx f_2(x_1, x_2, x_3).$$

Since all isomers of a minority multioperation are minority multioperations, it follows from the uniqueness of a minority multioperation $g \in \mathcal{C}$ that g is totally symmetric, that is invariant under all permutations of variables. This completes the proof of the first part of Lemma 2.2.

For simplicity, we write $(x_1 x_2 x_3)$ for $g(x_1, x_2, x_3)$.
From now, we suppose that $\mathcal{C}^{(4)} \setminus \mathcal{Q}^{(4)}$ contains no semiprojection. For fixed $a, b \in A$, define $\varphi_{ab} : A \to \mathfrak{P}(A)$ by $\varphi_{ab}(x) :\approx g(x, a, b)$.

Claim 2.8 *For all $a, b \in A$ the map $\varphi := \varphi_{ab}$ is a permutation of A of order at most 2 (i.e. an involution) and consequently $g \in \mathcal{O}^3$ (i.e. an operation).*

Proof of Claim 2.8. From (4) for $f_1 = f_2 = g$ we have

$$\varphi^2(x) = g(g(x, a, b), a, b) \approx \{x\}.$$

We show that $\varphi \in \mathcal{O}^{(1)}$, that is φ is a selfmap of A. Indeed, let $c \in A$ and $u \in \varphi(c)$ be arbitrary. Then $\varphi(u) \in \varphi^2(c) = \{c\}$ and thus $\{u\} = \varphi^2(u) = \varphi(c)$ proving that $|\varphi(c)| = 1$. It follows that φ is a permutation of A such that $\varphi^2 = e_1^1$. □

Claim 2.9 *Fix $0 \in A$ and put $x + y :\approx (xy0)$. Then $< A; +, 0 >$ is a Boolean group and $(xyz) \approx x + y + z$.*

Proof of Claim 2.9. From the total symmetry of ( ) and from $x00 \approx x$ we obtain that $< A; +, 0 >$ is a commutative groupoid with the neutral element 0. Next, $a + b = 0$ if and only if $a = b$. Indeed, if $a + b = 0$, then $(ab0) = 0$, hence $a = b$ by Claim 2.6. Conversely, we have $a + a = (aa0) = 0$.

We prove that $(xyz) \approx x + (y + z)$, that is, in view of the observation just above, $(xyz) + (x + (y + z)) \approx 0$. Using the definition of our groupoid

operation, this means $((xyz)(x(yz0)0)0) \approx 0$. It suffices then to prove that the following identity holds

$$((xyz)(x(yzt)t)t) \approx t. \tag{7}$$

Let h be the quaternary term operation on $< A; (\ ) >$ defined by :

$$h(x_1, x_2, x_3, x_4) :\approx ((x_1x_2x_3)(x_1(x_2x_3x_4)x_4)x_4)). \tag{8}$$

We first show that h is a semiprojection on its last variable. We abbreviate the right-hand side of (8) by ((123)(1(234)4)4). We consider the six possibilities of identifying two variables.

a). Set $x_1 = x_2$. Using the fact that $(\ )$ is a totally symmetric minority operation and (4) of Claim 2.5 (for $f_1 = f_2 = (\ )$), we get

$$((113)(1(134)4)4) = (3((314)14)4) = (3(1(134)4)4) = (334) = 4.$$

b). Set $x_1 = x_3$. Due to the total symmetry of $(\ )$ this case reduces to the previous one.

c). Set $x_1 = x_4$. Using the fact that $(\ )$ is a totally symmetric minority operation, we get

$$((423)(4(234)4)4) = ((423)(234)4) = ((234)(234)4) = 4.$$

d). Set $x_2 = x_3$. Using the fact that $(\ )$ is a minority operation we get

$$((122)(1(224)4)4) = (1(144)4) = (114) = 4.$$

e). Set $x_2 = x_4$. Using the fact that $(\ )$ is a totally symmetric minority operation we get

$$((143)(1(434)4)4) = ((143)(134)4) = ((134)(134)4) = 4.$$

f). Set $x_3 = x_4$. As above we get

$$((124)(1(244)4)4) = ((124)(124)4) = 4.$$

Since $\mathcal{C}^{(4)} \setminus \mathcal{Q}^{(4)}$ contains no semiprojection, h is a projection, in fact $h = e_4^4$, proving that identity (7) holds. From this identity and the total symmetry of $(\ )$ we get $x + (y + z) \approx (zxy) \approx (xyz) \approx x + (y + z)$ proving that the binary operation $+$ is associative. This concludes the proof of the claim. □

The proof of Lemma 2.2 is complete. □

3 Proof of Theorem 1.2

If G is a 2-elementary group and $\mathcal{C} = F_G$ (where F_G was introduced in Theorem 1.2) then, clearly, $\mathcal{C}$ contains all constant maps, all its binary idempotent members are projections and the term operation $x + y + z$ is idempotent and not a projection.
Conversely, suppose that $\mathcal{C}$ contains all constant operations, that its idempotent binary operations are the two projections and some n-ary idempotent operation is not a projection for a fixed $n \geq 3$. With the following lemma, we get that $\mathcal{C} \setminus \mathcal{Q}$ contains neither a semiprojection nor a majority operation.

Lemma 3.1 *Let $\mathcal{C}$ be a clone on a universe A with at least two elements, that contains all constant operations and such that the binary idempotent operations are the two projections. Then for all $n \geq 2$ an operation $g \in \mathcal{C}^{(n)}$ is a projection if an only if some isomer f of g satisfies*

$$f(y, y, x_3, \ldots, x_n) \approx y. \tag{9}$$

Proof. The proof is an adaptation of Lemma 2.4 of [7]. The necessity of (9) is obvious. We prove the sufficiency of (9) by induction on n ($n \geq 2$). For $n = 2$ if $f(y, y) \approx y$, f is idempotent and f is a projection by the hypothesis. Suppose $n \geq 3$ and every $h \in \mathcal{C}^{(n-1)}$ satisfying (9) is e_i^{n-1} for some $i \in \{1, 2\}$. Since $\mathcal{C}$ contains all constant operations, for every $a \in A$, it contains the $(n-1)$-ary operation f_a defined by $f_a(x_1, \ldots, x_{n-1}) :\approx f(x_1, \ldots, x_{n-1}, a)$. From the inductive hypothesis, $f(x_1, \ldots, x_{n-1}, a) \approx x_{i(a)}$ for some $i(a) \in \{1, 2\}$. If $i(a) = 2$ for all $a \in A$ then f is the second projection. Suppose $i(b) = 1$ for some $b \in A$. The $(n-1)$-ary operation f' defined by $f'(x_1, \ldots, x_{n-1}) :\approx f(x_1, \ldots, x_{n-1}, x_{n-1})$ belongs to $\mathcal{C}$ and satisfies the same hypothesis as f, hence it is either the first or the second projection. Since, from above, $f'(x_1, \ldots, x_{n-2}, b) \approx f(x_1, \ldots, x_{n-2}, b, b) \approx x_1$, clearly f' is the first projection. Let $a \in A$ be arbitrary. From $x_1 \approx f(x_1, \ldots, x_{n-2}, a, a) \approx x_{i(a)}$ we obtain $i(a) = 1$ proving that f is the first projection. This proves the inductive step and the lemma. $\square$

Let $\mathcal{C}' := \mathcal{C} \cap \mathcal{I}$ be the clone of the idempotent operations from $\mathcal{C}$. According to Theorem 1.1, $\mathcal{C}'^{(3)} \setminus \mathcal{Q}^{(3)} = \{m_G\}$ where $m_G(x, y, z) \approx x + y + z$ for a Boolean group $G :=< A; +, 0 >$. From [7] (the statement at the end of page 173) follows that $\mathcal{C} = F_G$. With this the proof of Theorem 1.2 is complete.

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